[4] Hosford, M. J. and Rosebrock, T. L. "The ballistic pendulum applied to pulsed plasma acceleration," Allison Div., General Motors Corp., EDR 2253 (March 1962).

Reply by Authors to D. L. Clingman and T. L. Rosebrock

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IN a previous paper,[1] the authors wanted to illustrate that the simple ballistic pendulum is subject to error, and effects such as were observed should be considered before accepting impulse data obtained with pendulums. It was not intended to condemn ballistic pendulums per se; in fact, it was hoped to apply this technique in the authors' present work.

The solution to the ablation problem proposed by D. L. Clingman and T. L. Rosebrock seems to be a good one. Another approach simply is to reduce the energy density of the plasma at the surface of the pendulum by moving the pendulum away from the plasma accelerator so that the beam can spread radially and axially; however, the pendulum must become larger to collect all of the beam.

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Comment on "Invariant Two-Body Velocity Components"

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IN a previous paper,[1] Cronin and Schwartz discussed the invariant two-body velocity components of orbital body motion. Speculation was presented on possible useful applications of this property of invariance to the solution of two-body trajectory problems.

Since about three years ago, various studies of such applications have been conducted and are continuing. As noted in a XIIIth International Astronautical Congress paper,[2] additional literature on such applications is available.[3-14] Also, Newton's paper[15] is based upon this invariance property.

In general, the invariance of the specified orbital velocity components describes a circle that defines the orbital velocity vector in inertial space. This figure is known as the velocity hodograph, originally discovered by Hamilton and Möbius.

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[13] Altman, S. P. and Pistiner, J. S., "Comment on the correlation of Stark's two-terminal trajectory optimization with an orbital hodograph analysis," ARS J. 32, 983 (1962).

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[15] Newton, R. R., "Variables that are determinate for any orbit," ARS J. 31, 364–366 (1961).

Explicit Solution of the "Three-Moments Equation"

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THE solution of the "three-moments equation" given by the author[1] requires the search for eigenvalues and eigenvectors of a certain operator. To remove these difficulties and to find the explicit solution, one can substitute, instead of a finite-difference operator and boundary conditions, a new operator that transforms the boundary-value problem into an algebraic equation, without previously solving the problem. Commutation properties of this operator with Green's function make it possible to find the solution in classical form.

The "three-moments problem" has the notation

$$L_{mn}M_n = -G_m$$

$$M_0 = M_N = 0 \qquad n = 0,1,\ldots,N \qquad (1)$$

where the finite-difference operator L and boundary conditions can be written as follows:

$$\left. \begin{array}{l} L_{mn} = \delta_{m+1,n} + 4\delta_{m,n} + \delta_{m-1,n} \\ M_0 = M_N = 0 \end{array} \right\} \quad n = 0,1,\ldots,N \quad (2)$$

[1] Wolff, T., "Direct solution of the 'three-moments equation'," AIAA J. 1, 718 (1963).

and corresponding equation for the Green's function in the form

$$L_{mk}G_{kn} = -\delta_{mn} \qquad (3)$$

The most popular method of solution of boundary-value problems like the foregoing is founded on Green's function. With its help, the moment on the nth support has the value

$$M_n = G_{nk}G_k \qquad (4)$$

To modify this method and to change the finite-difference operator **L** and boundary values (2) into single operator **R** or to change the boundary-value problem (1) into the algebraic equation

$$R_{mn}M_n = -G_m \qquad (5)$$

the operator **L** will be expanded. The eigenvalues can be found from the equation $|\mathbf{I} - \lambda \mathbf{L}| = 0$ or from the equivalent equation

$$[\delta_{m,n} - \lambda_k(\delta_{m,n+1} + 4\delta_{m,n} + \delta_{m,n-1})]\sin(k\pi m/N) = 0 \qquad (6)$$

where

$$\lambda_k = \frac{1}{2[2 + \cos(k\pi/N)]}$$

and $N - 1$ is the number of rows and columns in matrix **L**. But it is easy to prove that $\sin(k\pi n/N)$ represents the components of the eigenvectors of **L**. To carry out the normalization, these quantities must be multiplied by a constant factor, and then one gets the "projection tensors"

$$P_{m,n;k} = \frac{\sin\dfrac{k\pi m}{N}\sin\dfrac{k\pi n}{N}}{\displaystyle\sum_{n=0}^{N}\sin^2\left(\dfrac{k\pi n}{N}\right)} \qquad (7)$$

Thus one can write the operator **R**, which substitutes for (2) in expanded form,

$$\mathbf{R} = \frac{1}{\lambda_k}\mathbf{P}_k \qquad (8)$$

and for Eq. (5) as follows:

$$(1/\lambda_k)P_{m,n;k}M_n = -G_m \qquad (9)$$

From (9), with help of the tensor property of **P**,

$$\mathbf{P}_l\mathbf{P}_k = \delta_{lk} \qquad (10)$$

one gets, after multiplication from the left by $\mathbf{P}_l$, the explicit solution of the "three-moments equation:"

$$M_k = -\lambda_l P_{k,n;l}G_n \qquad (11)$$

The comparison of (11) with (4) yields Green's function

$$G_{nm} = -\lambda_l P_{nm;l} \qquad l = 0,1,\ldots.N \qquad (12)$$

The comparison of (12) and (8) reveals that **R** and **G** are commuting operators. This property distinguishes the operators **R** and (2). After summation, it is possible to write Green's function (12) in the form

$$W_{nk} = -\lambda_l P_{n,k;l} = \frac{(-1)^{n+k}}{2\sinh\sigma}\left\{\sinh|n-k|\sigma - \right.$$
$$\left. \sinh(n+k)\sigma + \frac{2\sinh n\sigma \sinh k\sigma}{\tanh N\sigma}\right\} \qquad (13)$$

where $\sinh\sigma = 3^{1/2}$. Then the explicit solution (11) takes more compact form:

$$M_n = W_{nm}G_m \qquad (14)$$

Adding to (14) regular part, one can write the solution of the boundary value problem

$$M_{n+1} + 4M_n + M_{n-1} = -G_n$$
$$M_0 = M_0 \qquad M_N = M_N \qquad n = 0,1,\ldots N \qquad (15)$$

as follows:

$$M_n = W_{nk}G_k - (-1)^{N+n}(\sinh n\sigma/\sinh N\sigma) \times$$
$$[(-1)^N e^{N\sigma}M_0 - M_N] + (-1)^n e^{n\sigma}M_0 \qquad (16)$$

To facilitate insight into the structure of Green's function (12) and (13), one can take, as in the previous case, $N = 3$. Then $\lambda_1 = \frac{1}{5}$, $\lambda_2 = \frac{1}{3}$, and

$$G_{nm} = -\left(\frac{1}{5}\frac{\sin\dfrac{\pi m}{3}\sin\dfrac{\pi n}{3}}{\displaystyle\sum_{n=0}^{3}\sin^2\left(\dfrac{\pi n}{3}\right)} + \frac{1}{3}\frac{\sin\dfrac{2\pi m}{3}\sin\dfrac{2\pi n}{3}}{\displaystyle\sum_{n=0}^{3}\sin^2\left(\dfrac{2\pi n}{3}\right)}\right)$$

For particular numbers

$$\mathbf{G} = -\begin{vmatrix} \frac{1}{10} + \frac{1}{6}, & \frac{1}{10} - \frac{1}{6} \\ \frac{1}{10} - \frac{1}{6}, & \frac{1}{10} + \frac{1}{6} \end{vmatrix}$$

Assuming, for example, $N = 3$ and the load terms, $G_1 = G_2 = \frac{1}{2}pl^2$, one obtains from (15) the following expression for moments:

$$M_1 = M_2 = \frac{1}{2\sinh\sigma}\left\{-\sinh\sigma + 3\sinh 3\sigma - \right.$$
$$\sinh 4\sigma + \frac{2\sinh\sigma}{\tanh 3\sigma}\cdot[\sinh\sigma - \sinh 2\sigma +$$
$$\left. \sinh 3\sigma]\right\}\frac{1}{2}pl^2 = -\frac{1}{10}pl^2$$

in agreement with previous results.

Test Time in a 1.5-Inch-Diameter High-Stagnation-Enthalpy Shock Tube

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IN a recent article, Hoshizaki[1] questioned the validity of heat transfer measurements[2] made at high-stagnation-enthalpy conditions in a 1.5-in.-diam shock tube. The point in question is the available testing time in low pressure, high shock Mach number flow in a small diameter shock tube. The present note on test time measurements in the 1.5-in.-diam shock tube appears necessary to establish the validity of these measurements. Since the measurements proved to be quite accurate compared with Hoshizaki's theory and experimental results (as well as more recent data obtained by Rose and Stankevics[3] at the Avco Everett Laboratory), it would be expected that the test time was adequate.

The original heat transfer measurements relied on the heat transfer calorimeter gage to indicate test time. Camm and Rose[4] recently found that calorimeter measurements indicate steady test times that are longer than those determined by other methods. Apparently, the heat transfer is not changed by the change in composition of the flow at the interface. Since heat transfer is proportional mainly to the enthapy and density rather than temperature or composition, Camm and Rose suggest that heat transfer is unchanged by trace im-